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NON-LOCAL ANALYSIS OF FAMILIES OF PERIODIC SOLUTIONS IN AUTONOMOUS SYSTEMS†

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One-parameter families of periodic solutions arising from equilibrium positions of an autonomous system are considered. It is shown that they may be divided into families of the first and second kind; families of one kind cannot be identical when continued as the parameter is varied. As a result, a lower bound is obtained for the number of families that may be continued to arbitrary large values of the norm or the period, and an estimate is also obtained for the number of periodic solutions with a given minimal period. Additional properties of these families are established for Hamiltonian systems satisfying certain symmetry conditions. The results are illustrated for an articulated pendulum. © 2000 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

Consider an autonomous system depending on a parameter λ

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\lambda), \quad \mathbf{x}\in R^n, \quad \mathbf{f}(\mathbf{x},\lambda)\in C^3(R^n \times R, R^n) \tag{1.1}$$

and also a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}\in R^n, \quad \mathbf{f}(\mathbf{x})\in C^3(R^n, R^n) \tag{1.2}$$

which admits of a single-valued integral

$$H(\mathbf{x}(t)) \equiv \lambda = \text{const} \tag{1.3}$$

System (1.2), (1.3) may be reduced to the form of (1.1) by eliminating one of the variables through the use of integral (1.3). Conversely, introducing an additional variable $x_{n+1} = \lambda$ in (1.1) and adding the equation $\dot{x}_{n+1} = 0$, we obtain system (1.2), (1.3) with $H(\mathbf{x}) = x_{n+1}$. Note, however, that the results obtained below are directly applicable to either of these systems.

Among the systems of type (1.2), (1.3) there are, in particular, Lyapunov systems and Hamiltonian systems (in the latter case, some results analogous to those obtained below have been established before [1]).

Let $\mathbf{x}(\lambda)$ be the family of equilibrium positions of system (1.1) ($\mathbf{f}(\mathbf{x}(\lambda), \lambda) = 0$). If some $\lambda = \lambda_k$ exists such that the matrix $(\mathbf{f}_\mathbf{x}(\mathbf{x}(\lambda_k), \lambda_k)$ has a pair of pure imaginary eigenvalues $i\omega$, the other eigenvalues are not multiples of $i\omega$ and $\alpha'(\lambda_k) \neq 0$ ($\alpha(\lambda) + i\omega(\lambda)$ is the continuation as λ is varied of the eigenvalue $i\omega$), then, by Hopf's theorem [2], for small values of the parameter s a unique family of periodic solutions $\mathbf{x}(t, \lambda(s)) = \mathbf{x}(t + T(s), \lambda(s))$ exists such that

$$\mathbf{x}(t,\lambda(s)) \to \mathbf{x}_k = \mathbf{x}(\lambda_k), \quad T(s) \to 2\pi/\omega, \quad \lambda(s) \to \lambda_k \ \text{ as } \ s \to 0 \tag{1.4}$$

As a rule, the equilibrium positions $\mathbf{x}_k$ in system (1.2), (1.3) are isolated. By Lyapunov's theorem [3], another sufficient condition for the existence of families of periodic solutions of type (1.4) is that no eigenvalues exist which are multiples of $i\omega$. Note that this condition, like the condition $\alpha'(\lambda_k) \neq 0$ in Hopf's theorem, is satisfied in the generic case (which will be that considered below).

The aim of this paper is to make a global analysis of system (1.4). The following results of [4, 5] will be used.

Solutions may bifurcate as the parameter s is increased, so that continuation as s is varied may fail to be unique if s is sufficiently large. Nevertheless, from every "tree" of such solutions one can single out a one-parameter family $\mathbf{x}(t, \lambda(s))$ which has the following properties.

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Continued as s is varied, the family $\mathbf{x}(t, \lambda(s))$ may end at some equilibrium position $\mathbf{x}_q$ (in that case one may assume that $\mathbf{x}(t, \lambda(s))$ coincides with some family arising from $\mathbf{x}_q$; it may also happen that $q = k$). Otherwise, $\mathbf{x}(t, \lambda(s))$ may be continued to as large a value as desired

$$M(s) = T(s) + \| \mathbf{x}(s) \|, \quad \| \mathbf{x}(s) \| = | \lambda(s) | + \max_t \| \mathbf{x}(t, \lambda(s)) \| \tag{1.5}$$

The continuous function $T(s)$ is equal to the minimum period $T_{\min}(s)$ of the solution $\mathbf{x}(t, \lambda(s))$ with the possible exclusion of an at most denumerable set of points s_r where $T_{\min}(s_r) = T(s_r)/q_r$, where $q_r > 1$ is an integer (at such points families with minimum period $T(s_r)/q_r$ may bifurcate from $\mathbf{x}(t, \lambda(s))$).

The solution $\mathbf{x}(t, \lambda(s))$ of Eq. (1.1) or Eq. (1.2) corresponds to the variational equation

$$\dot{\mathbf{y}} = A(t, s)\mathbf{y}, \quad A(t, s) = \mathbf{f}_\mathbf{x}(\mathbf{x}(t, \lambda(s))) \tag{1.6}$$

Let $\rho_k(s)$ ($k = 1, \ldots, n$) denote the multipliers of Eq. (1.6). Since we are considering autonomous systems, this equation has a periodic solution $\mathbf{y}_1 = \dot{\mathbf{x}}(t, \lambda(s))$; corresponding to which is the multiplier $\rho_1(s) = 1$. In the event of an integral (1.3), the solution will correspond to a double multiplier $\rho_1(s) = \rho_2(s) = 1$ [6].

Fixing a value of some coordinate x_k, let us consider the corresponding Poincaré map $G(\mathbf{v}, \lambda)$. Suppose the orbit $\mathbf{x}(t, \lambda(s))$ intersects a plane $x_k = C$ at a point $\mathbf{v}(s)$; then

$$\mathbf{v}(s) = G(\mathbf{v}(s), \quad \lambda(s)) \tag{1.7}$$

As we know, the eigenvalues of the Jacobian $B(s) = G_\mathbf{v}(\mathbf{v}(s), \lambda(s))$ are $\rho_k(s), \ldots, \rho_n(s)$ ($k = 2$ for system (1.1) and $k = 3$ for system (1.2), (1.3)).

Set $\lambda_s(s) = d\lambda(s)/ds$ and $d(s) = \det[I - B(s)]$, where I is the identity matrix. It is obvious that

$$d(s) = (1 - \rho_k(s))(1 - \rho_{k+1}(s)) \ldots (1 - \rho_n(s)) \tag{1.8}$$

If $d(s) \neq 0$ then, by the Implicit Function Theorem, the solution $\mathbf{v}(s)$ of Eq. (1.7) is uniquely continuable as the parameter λ is varied, provided that $|\lambda - \lambda(s)|$ is small enough; hence $\lambda_s(s) \neq 0$. If $d(s_k) = 0$, then $\lambda_s(s_k) = 0$ [4], and conversely. Thus, the zeros of the functions $\lambda_s(s)$ and $d(s)$ coincide. In the generic case, all the zeros are simple.

2. GLOBAL CONTINUATION OF THE SOLUTION

Suppose the family $\mathbf{x}(t, \lambda(s)) = \mathbf{x}(t + T(s), \lambda(s))$ arises from a certain equilibrium condition x_k. We will denote by a_k the number of positive eigen values of the matrix $\mathbf{f}_\mathbf{x}(\mathbf{x}_k)$.

Definition. We will call $\mathbf{x}(t, \lambda(s))$ a family of the first (second) kind if the function $(-1)^{a_k}\lambda(s)$ increases (decreases) for small s.

Theorem 1. Families of one kind cannot coincide when continued with respect to a parameter.

Proof. Suppose the families $\mathbf{x}_1(t, \lambda_1(s))$ and $\mathbf{x}_2(t, \lambda_2(s))$, arising from equilibrium positions $\mathbf{x}_k$ and $\mathbf{x}_p$ (possible $k = p$), coincide. Then $\mathbf{x}_1(t, \lambda_1(s)) = \mathbf{x}_2(t, \lambda_2(s_* - s))$, $\mathbf{x}_1(t, \lambda_1(s_*)) = \mathbf{x}_p$ and $\lambda_1(s) = \lambda_2(s_* - s)$ for some s^*. Suppose one of $\lambda_1(s)$, $\lambda_2(s)$ is an increasing function and the other is a decreasing function for small s; then $d\lambda_1(s)/ds$ and, consequently, $d(s)$ has an even number of zeros in $(0, s_*)$; consequently, $d(0)$ and $d(s_*)$ have the same signs. As can be seen from (1.8), this happens if the number of multipliers $\rho_q \in (1, \infty)$ when $s = 0$ and $s = s_*$ is either even at both points or odd at both points (complex multipliers are conjugate in pairs and do not affect the sign of $d(s)$). Such multipliers correspond to positive eigenvalues μ_q and ν_q of the matrices $\mathbf{f}_\mathbf{x}(\mathbf{x}_k)$ and $\mathbf{f}_\mathbf{x}(\mathbf{x}_p)$ ($\rho_q(0) = \exp(\mu_q T_1(0))$ and $\rho_q(s_*) = \exp(\nu_q T_1(s_*))$; therefore, the quantities a_k and a_p are either both even or both odd. Since, by assumption, one of the functions $\lambda_1(s)$, $\lambda_2(s)$ increases and the other decreases for small s, it follows that the families $\mathbf{x}_1(t, \lambda_1(s))$ and $\mathbf{x}_2(t, \lambda_2(s))$ are of different kinds.

But if the functions $\lambda_1(s)$, $\lambda_2(s)$ vary in the same direction, then $d(s)$ has an odd number of zeros in $(0, s_*)$. Consequently, one of the quantities a_k, a_p is even and the other odd, so that the families are of different kinds.

Thus, in all cases, families that coincide when continued as s is varied are of different kinds. The theorem is proved.

Let us assume that the system has a finite number of equilibrium positions $\mathbf{x}_k$, $\lambda_k (k = 1, ..., p)$, and therefore a finite number of corresponding families of periodic solutions. Let N_1 and N_2 denote the total number of families of the first and second kinds respectively. As follows from Theorem 1, when they are continued as the parameter is varied, only families of different kinds may coincide. Hence the following proposition holds.

Corollary 1. At least $|N_1 - N_2|$ families of periodic solutions may be continued as the parameter s is varied up to values of M as large as desired.

It is obvious that families $\mathbf{x}_1(t, \lambda_1(s))$ and $\mathbf{x}_2(t, \lambda_2(s))$ arising from a single equilibrium position are of the same kind if $\lambda_1(s)$ and $\lambda_2(s)$ vary in the same sense for small s, i.e., if the corresponding bifurcations are of the same type (supercritical or subcritical). Therefore, in a system with one equilibrium position, the number of families that are continuable as s is varied up to values of M as large as desired is at least $|n_1 - n_2|$, where n_1 and n_2 are the number of supercritical and subcritical bifurcations.

Remark. It has been proved [4] that an equilibrium point $\mathbf{x}_k$, λ_k to which one family of periodic solutions corresponds may be assigned the index +1 ("source") or −1 ("sink") depending on the sign of $\alpha'(\lambda_k)$ $(\alpha(\lambda_k) = 0)$ and the number a_k of positive eigenvalues of the matrix $\mathbf{f}_\mathbf{x}(\mathbf{x}_k)$. This proves that families corresponding to equilibrium positions with different indices cannot coincide when continued as the parameter is varied. A classification of this kind is not applicable to system (1.2), (1.3), because here the equilibrium positions are usually isolated (whereas in system (1.1) the functions $\alpha(\lambda)$ corresponds to a continuum of equilibrium positions $\mathbf{x}(\lambda)$); in addition, the system may have several families of periodic solutions. Hence the classification assumed in this paper, which is equally applicable to systems (1.1) and (1.2), (1.3), is in this sense more general.

3. THE EXISTENCE OF SOLUTIONS WITH A GIVEN PERIOD

We now consider the problem of whether systems (1.1) and (1.2), (1.3) have periodic solutions with a given minimum period T. Incidentally, known results of this kind are concerned mainly with Hamiltonian systems (see, e.g., the survey [7]).

Without loss of generality, we will assume that in system (1.1) $\mathbf{x}_k = 0$, $\lambda_k = 0$ for some k, in which case $\mathbf{f}(\mathbf{0}, 0) = \mathbf{0}$. Let us assume that for suitable constants L, C and r

$$\|\mathbf{f}(\mathbf{x}, \lambda)\| < L\|\mathbf{x}\| + C \quad \text{for} \quad \|\mathbf{x}\| + |\lambda| > r \tag{3.1}$$

As to Eq. (1.2), we may assume that $\mathbf{f}(\mathbf{0})$, so that here too we assume that

$$\|\mathbf{f}(\mathbf{x})\| < L\|\mathbf{x}\| \quad \text{for} \quad \|\mathbf{x}\| > r \tag{3.2}$$

Let $N_1(T)$ and $N_2(T)$ denote the number of families $\mathbf{x}_k(t, \lambda(s))$ of the first and second kind satisfying the additional condition $T_k(0) = 2\pi/\omega_k < T$.

Theorem 2. For almost all $T < 2\pi/L$ at least $|N_1(T) - N_2(T)|$ periodic solutions exist with minimum period T.

Proof. As is obvious from the proof of Theorem 1, at least $|N_1(T) - N_2(T)|$ families of periodic solutions $\mathbf{x}_k(t, \lambda(s))$ with initial periods $T_k(0) < T$ cannot coincide with one another when continued as s is varied. Consequently, such a family either coincides with a family of another kind $\mathbf{x}_p(t, \lambda(s))$ whose initial period is $T_p(0) > T$ or is continuable as s is varied up to values of $T_k(s)$ or $\|\mathbf{x}_k(s)\|$ as large as desired. It is clear that in the first two cases $T_k(s) = T$ for some s; we will show that this happens in the last case as well.

By (3.1), constants L' and C' exist such that, for all $\lambda \in R$ and $\mathbf{x} \in R^n$

$$\|\mathbf{f}(\mathbf{x}, \lambda)\| < L'\|\mathbf{x}\| + C' \tag{3.3}$$

Consequently

$$d\|\mathbf{x}\|/dt = (\mathbf{x}, \dot{\mathbf{x}})/\|\mathbf{x}\| \leqslant \|\dot{\mathbf{x}}\| = \|\mathbf{f}(\mathbf{x}, \lambda)\| \leqslant L'\|\mathbf{x}\| + C' \tag{3.4}$$

Therefore, for any solution $\mathbf{x}(t)$ of Eq. (1.1) the following inequality holds

$$\|\mathbf{x}(t)\| \leqslant (\|\mathbf{x}(t_0)\| + C'/L')\exp[L'(t - t_0)] - C'/L' \tag{3.5}$$

Suppose

$$\max_t \|\mathbf{x}_k(t,\ \lambda(s))\| \to \infty \quad \text{as} \quad s \to \infty$$

but $T_k(s)$ and $\lambda(s)$ remain bounded. Since the minimum distance between extremal values of $\|\mathbf{x}_k(t, \lambda(s))\|$ is less than $T_k(s)$, it follows by (3.5) that $\|\mathbf{x}_k(t, \lambda(s))\| \to \infty$ for all t, and therefore $\|\mathbf{x}_k(t, \lambda(s))\| > r$ for large s. But then, by a theorem due to Yorke [8], if condition (3.1) holds, we have $T_k(s) > 2\pi/L > T$; consequently, $T_k(s) = T$ for some s. Clearly, this conclusion holds if $\lambda(s) \to \infty$ as $s \to \infty$.

In system (1.2), (1.3) the quantity $\lambda(s) = H(\mathbf{x}(s))$ remains bounded for finite $\|\mathbf{x}(s)\|$; hence $\|\mathbf{x}(s)\| \to \infty$ as $M(s) \to \infty$, so that the above arguments remain valid.

As remarked in the introduction, $T_k(s)$ is equal, for almost all s, to the minimum period of the solution $\mathbf{x}_k(t, s)$. The theorem is proved.

If

$$\lim(\|\mathbf{f}(\mathbf{x}, \lambda)\| / \|\mathbf{x}\|) \to 0 \quad \text{as} \quad \|\mathbf{x}\| + |\lambda| \to \infty$$

then the constant L in (3.1) may be assumed to be as small as desired. Hence the theorem is true for any T. In particular, in the case of a single equilibrium position, the number of T-periodic solutions is at least $|n_1(T) - n_2(T)|$, where $n_1(T)$ and $n_2(T)$ are the numbers of supercritical and subcritical bifurcations with initial periods less than T.

4. HAMILTONIAN SYSTEMS

Let us consider in greater detail Hamiltonian systems

$$\dot{\mathbf{x}} = JH_{\mathbf{x}}(\mathbf{x}), \quad J = \begin{Vmatrix} 0 & I \\ -I & 0 \end{Vmatrix}, \quad \mathbf{x} \in R^{2n} \tag{4.1}$$

Equation (4.1) admits of an integral (1.3), and so the results obtained above remain valid. It is obvious that for a family $\mathbf{x}(t, \lambda(s))$ corresponding to an equilibrium position $\mathbf{x}_k$, the energy $\lambda(s)$ of the system varies in the same sense as for the corresponding family $\mathbf{x}^0(t, s)$ of the linearized system ($H^0(\mathbf{x}) = (H_{\mathbf{xx}}(\mathbf{x}_k)\mathbf{x}, \mathbf{x})$, where $(\mathbf{a}, \mathbf{b})$ denotes the scalar product of vectors $\mathbf{a}$ and $\mathbf{b}$). Assuming without loss of generality that $\mathbf{x}_k = 0$, we take $\mathbf{x}^0(t, s) = s\mathbf{x}(t)$, so that

$$\lambda_s(0) = 2s(H_{\mathbf{xx}}(\mathbf{0})\mathbf{x}(t),\ \mathbf{x}(t))$$

We know that in Hamiltonian systems with a sign-definite Hessian $H_{\mathbf{xx}}(\mathbf{0})$, all the eigenvalues of the matrix $JH_{\mathbf{xx}}(\mathbf{0})$ are pure imaginary, so that $a_k = 0$. As a result, if $H_{\mathbf{xx}}(\mathbf{0}) > 0$ all families are of the first kind ($\lambda_s(0) > 0$), and, if $H_{\mathbf{xx}}(\mathbf{0}) < 0$, they are all of the second kind ($\lambda_s(0) < 0$). If the Hessian $H_{\mathbf{xx}}(\mathbf{0})$ is not sign-definite, the system may have families of different types.

Let us assume that the linearized system has the form

$$\dot{\mathbf{x}} = JS\mathbf{x}, \quad S = \begin{Vmatrix} C & 0 \\ 0 & L \end{Vmatrix} \tag{4.2}$$

where C and L are symmetric matrices of order n, L being positive-definite. In particular, the system

$$M\ddot{\mathbf{z}} + C\mathbf{z} = \mathbf{0}, \quad \mathbf{z} \in R^n, \quad M > 0 \tag{4.3}$$

may be reduced to the form (4.2) if we put $\mathbf{z} = \mathbf{x}_1$, $M\dot{\mathbf{z}} = \mathbf{x}_2$, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, $L = M^{-1}$.

For an oscillatory solution of Eq. (4.2), $\mathbf{x} = \mathbf{c}\exp(i\omega t)$ ($\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2)$), we have $i\omega\mathbf{c}_1 = L\mathbf{c}_2$, $-i\omega\mathbf{c}_2 = C\mathbf{c}_1$, and therefore

$$(S\mathbf{x},\ \mathbf{x}) = (C\mathbf{c}_1,\ \mathbf{c}_1) + (L\mathbf{c}_2,\ \mathbf{c}_2) = 2(L\mathbf{c}_2,\ \mathbf{c}_2) > 0 \quad (L > 0)$$

Thus, in the system under consideration, $\lambda_s(0) > 0$ for any family.

In applications one often encounters Hamiltonians satisfying certain symmetry conditions; we will show that the families of solutions being considered here will then possess certain additional properties.

Let us first assume that

$$H(\mathbf{q}, \mathbf{p}) = H(\mathbf{q}, -\mathbf{p}) \tag{4.4}$$

where $\mathbf{q} = (x_1, \ldots, x_n)$ and $\mathbf{p} = (x_{n+1}, \ldots, x_{2n})$ are generalized coordinates and momenta. A relationship of this kind holds, in particular, when the Hamiltonian is the sum of the kinetic and potential energies of a mechanical system; we then have

$$H(\mathbf{q}, \mathbf{p}) = \sum_{p,k=1}^{n} a_{pk}(\mathbf{q}) p_p p_k + V(\mathbf{q}) \tag{4.5}$$

Proposition 1. In system (4.1), (4.4) the family $\mathbf{x}(t, s) = (\mathbf{q}(t, s), \mathbf{p}(t, s))$ will satisfy the following relations if the origin of t is suitably chosen

$$\mathbf{q}(t, s) = \mathbf{q}(-t, s), \quad \mathbf{p}(t, s) = -\mathbf{p}(-t, s) \tag{4.6}$$

Proof. By (4.4)

$$H_{\mathbf{q}}(\mathbf{q}, \mathbf{p}) = H_{\mathbf{q}}(\mathbf{q}, -\mathbf{p}), \quad H_{\mathbf{p}}(\mathbf{q}, \mathbf{p}) = -H_{\mathbf{p}}(\mathbf{q}, -\mathbf{p}) \tag{4.7}$$

Therefore, together with $\mathbf{x}(t, s)$, the function $\mathbf{x}_*(t, s) = (\mathbf{q}(-t, s), -\mathbf{p}(-t, s))$ is also a solution of system (4.1). Since the solution $\mathbf{x}(t, s)$ for sufficiently small is unique up to a linear translation of t, it follows that $\mathbf{x}_*(t, s) = \mathbf{x}(t + l, s)$ for some l. Applying the transformation $t \to t + l/2$, we find that the solution $\mathbf{x}(t, s)$ satisfies (4.6). If this relation fails to hold for some s_*, then for $s > s_*$ a family $\mathbf{x}_*(t, s)$ also exists with the same minimum period $T(s)$, which coincides with $\mathbf{x}(t, s)$ when $s = s_*$. But this is impossible, since in the generic case the minimum period of the families bifurcating from $\mathbf{x}(t, s)$ is close to $T(s)/q$, where $q > 1$, in the neighbourhood of the bifurcation point [4].

Let us assume now that

$$H(\mathbf{x}) = H(-\mathbf{x}) \tag{4.8}$$

In particular, this condition will hold for Hamiltonian (4.5) if $a_{pk}(\mathbf{q}) = a_{pk}(-\mathbf{q})(p, k = 1, \ldots, n)$ and $V(\mathbf{q}) = V(-\mathbf{q})$.

By (4.8), $H_{\mathbf{x}}(\mathbf{0}) = \mathbf{0}$, that is, $\mathbf{x} = \mathbf{0}$ is an equilibrium point of the system; let $\mathbf{x}(t, s)$ be the corresponding family of periodic solutions.

Proposition 2. If condition (4.8) holds, the family $\mathbf{x}(t, s)$ satisfies the relation

$$\mathbf{x}(t, s) = -\mathbf{x}(t + T/2, s) \tag{4.9}$$

Proof. By condition (4.8), the function $-\mathbf{x}(t, s)$ also satisfies Eq. (4.1). For small s, by virtue of the uniqueness of the solution, we have $-\mathbf{x}(t, s) = \mathbf{x}(t + l, s)$ for some $l < T$. Consequently, $\mathbf{x}(t + 2l, s) = \mathbf{x}(t, s)$, that is, $l = T/2$. The rest of the proof is analogous to that of Proposition 1.

Corollary 2. If condition (4.8) holds, the family $\mathbf{x}(t, s)$ cannot end at an equilibrium point $\mathbf{x}_k \neq \mathbf{0}$.

Indeed, by virtue of (4.9), the mean value of $\mathbf{x}(t, s)$ in $(0, T(s))$ is zero, which is obviously not true for solutions close to $\mathbf{x}_k \neq \mathbf{0}$.

Note that the above arguments do not imply that the systems under consideration have no solutions other than those satisfying relations (4.6) or (4.9); rather, such solutions cannot be obtained by continuation of Lyapunov families as the parameter is varied.

5. EXAMPLE

The previous results will now be illustrated by an example: the free oscillations of an n-articulated pendulum (see Fig. 1). The kinetic and potential energies of the system are

$$K = \frac{1}{2} M_i \{[\sum_{k=i}^{s} l_k \dot{x}_k \sin x_k]^2 + [\sum_{k=i}^{s} l_k \dot{x}_k \cos x_k]^2\} \tag{5.1}$$

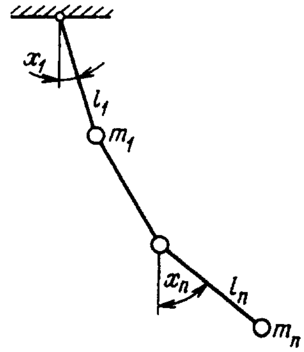

Fig. 1.

$$V = g\sum_{j=i}^{n} [l_j(1-\cos x_j)M_s]; \quad M_s = m_s + m_{s+1} + \ldots + m_n$$

where x_s, m_s and l_s are the angular coordinates, masses and lengths of the sections, and g is the acceleration due to gravity.

The system has an integral

$$K(\mathbf{x}(t), \dot{\mathbf{x}}(t)) + V(\mathbf{x}(t)) \equiv \lambda \tag{5.2}$$

In an equilibrium position $\mathbf{x}_k = (x_1^k, \ldots, x_n^k)$, the coordinates are $x_q^k = 0$ or π. Hence, the total number of equilibrium positions is 2^n. Linearized in the neighbourhood of $\mathbf{x}_k$, the system has the form of (4.3), where

$$C = \operatorname{diag}(c_1^k, \ldots, c_n^k), \quad c_s^k = gl_s \cos x_s^k M_s$$
$$M = [m_{pq}]_1^n, \quad m_{pq} = l_p l_q \cos x_p^k \cos x_q^k M_q \quad (q \geqslant p), \quad m_{qp} = m_{pq} \tag{5.3}$$

Since $M > 0$, the number m of positive eigenvalues of the matrix $M^{-1}C$ equals the number of positive c_s^k, that is, the number of coordinates $x_s^k = 0$. In the generic case, for each such quantity $(\omega_q^k)^2$ there is a family of periodic solutions of the original system $\mathbf{x}_q^k(t,s)(\mathbf{x}_q^k(t,s) \to \mathbf{x}_k$, $T_q^k(s) \to 2\pi/\omega_q^k$ as $s \to 0)$.

The number of equilibrium positions for a given m equals the number of combinations C_n^m; consequently, the total number of these families is $N = C_n^1 + 2C_n^2 + \ldots + nC_n^n$.

It is obvious that in $\mathbf{z} = \mathbf{x} - \mathbf{x}_k$ coordinates the kinetic and potential energies of the system satisfy the relations

$$K(\mathbf{z}, \dot{\mathbf{z}}) = K(\mathbf{z}, -\dot{\mathbf{z}}), \quad K(\mathbf{z}, \dot{\mathbf{z}}) = K(-\mathbf{z}, -\dot{\mathbf{z}}), \quad V(\mathbf{z}) = V(-\mathbf{z})$$

Therefore, the corresponding Hamiltonian $H(\mathbf{q}, \mathbf{p})$ satisfies conditions (4.4) and (4.8). As a result, the Lyapunov families satisfy relations (4.6) and (4.9). Returning to the original coordinates, we obtain

$$\mathbf{x}_q^k(t, s) = \mathbf{x}_q^k(-t, s), \quad \dot{\mathbf{x}}_q^k(t, s) = -\dot{\mathbf{x}}_q^k(-t, s)$$
$$\mathbf{x}_q^k(t, s) - \mathbf{x}_k = -\mathbf{x}_q^k(t + T/2, s) + \mathbf{x}_k \tag{5.4}$$

Since the linearized system has the form (4.3) it follows, as shown previously, that for all families $\lambda_s(0) > 0$. Hence, families that arise from the same equilibrium position are of the same kind (first or second, if the number of quantities $a_k = n - m_k$ is even or odd, respectively). Consequently, such families cannot coincide when continued by varying the parameter. On the other hand, by the first and third relations in (5.4), $\mathbf{x}_q^k(T/4, s) = \mathbf{x}_k$, so that families emanating from different equilibrium positions $\mathbf{x}_k$ and $\mathbf{x}_r$ also do not coincide. Thus, every family is continuable up to norms or periods as large as desired.

We will show that in any case $T_q^k(s) \to \infty$ as $s \to \infty$.

Indeed, as is obvious from (5.1)

$$K(\mathbf{x},\ \dot{\mathbf{x}}) \geqslant 0,\quad 0 \leqslant V(\mathbf{x}) \leqslant V_* = 2g\sum_{s=1}^{n} l_s M_s$$

By the first relation of (5.4)

$$\dot{\mathbf{x}}_q^k(0,\ s) = \mathbf{0},\quad K(\mathbf{x}_q^k(0,\ s),\ \dot{\mathbf{x}}_q^k(0,\ s)) = 0$$

Taking (5.2) into consideration, we find that $\lambda \leqslant V_*$, and so

$$K(\mathbf{x}_q^k(t,\ s),\ \dot{\mathbf{x}}_q^k(t,\ s)) \leqslant V_*$$

for all t. Since $K(\mathbf{x}, \dot{\mathbf{x}})$ is a positive-definite quadratic form in $\dot{\mathbf{x}}$ whose coefficients are periodic functions of $\mathbf{x}$, it follows from the last inequality that

$$\|\dot{\mathbf{x}}_q^k(t,\ s)\| \leqslant c$$

for some constant c and any t and s. Therefore, for any coordinate, we have

$$|x_{qp}^k(t,s) - x_{qp}^k(T/4)| \leqslant c\,|t - T/4|$$

Consequently, the period $T_q^k(s)$ tends to infinity together with $\|\mathbf{x}_q^k(s)\|$.

Let ω_* be the least frequency of free oscillations of the systems linearized at $\mathbf{x} = \mathbf{x}_k (k = 1, \ldots, 2^n)$. It follows from the preceding results that for any $T > 2\pi/\omega_*$ at least N periodic solutions of type (5.4) exist with the least period T.

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